

A Note on Holographic Renormalization of Probe D-Branes

PAOLO BENINCASA

Center for Particle Theory & Department of Mathematical Sciences
Science Laboratories, South Road, Durham DH1 3LE, United Kingdom

paolo.benincasa@durham.ac.uk

Abstract

A great deal of progress has been recently made in the study of holography for non-conformal branes. Considering the near-horizon limit of backgrounds generated by such branes, we discuss the holographic renormalization of probe D-branes in these geometries. More specifically, we discuss in some detail systems with a codimension-one defect. For this class of systems, the mode which describes the probe branes wrapping a maximal S^2 in the transverse space behaves like a free massive scalar propagating in a higher-dimensional (asymptotically) AdS_{q+1} -space. The counterterms needed are then the ones of a free massive scalar in asymptotically AdS_{q+1} . The original problem can be recovered by compactifying the asymptotically AdS -space on a torus and finally performing the analytic continuation of q to the value of interest, which can be fractional. We compute the one-point correlator for the operator dual to the embedding function. We finally comment on holographic renormalization in the more general cases of codimension- k defects ($k = 0, 1, 2$). In all the cases the embedding function exhibits the behaviour of a free massive scalar in an AdS -space and, therefore, the procedure outlined before can be straightforwardly applied. Our analysis completes the discussion of holographic renormalization of probe D-branes started by Karch, O'Bannon and Skenderis.

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1 Introduction

Gauge/gravity correspondence [1, 2, 3, 4] provides a powerful tool to investigate the dynamics of strongly coupled gauge theories. The original formulation [1] conjectures the equivalence between supergravity on the “near-horizon” geometry generated by a stack of N coincident D3-branes ($AdS_5 \times S^5$) and the gauge theory (four-dimensional $\mathcal{N} = 4$ $SU(N)$ Supersymmetric Yang-Mills) living on the boundary of AdS_5 , which describes the brane modes decoupled from the bulk. It can be straightforwardly extended to any asymptotically $AdS \times \mathcal{M}$ geometry, \mathcal{M} being a compact manifold. This conjectured equivalence is made precise by identifying the string partition function with the generating function for the gauge theory correlators, with the boundary value of the bulk modes acting as source of the correspondent gauge theory operator.

An important issue is the presence of divergences on both sides of the correspondence: the divergences appear in the UV-region on the gauge theory side and in the IR-region on the gravity side. These two divergence structures are related to each other [5], which can be intuitively understood by thinking that, on the gravity side, going to the IR-region means approaching the boundary (where the gauge theory sits). Holographic renormalization consistently deals with these IR divergences [6, 7, 8, 9, 10, 11, 12, 13, 14, 15] so that physical quantities can be computed. The first step is to find a solution for the bulk fields in a neighbourhood of the boundary and regularize the action by means of an IR-regulator. By inserting the solution previously found in the action, one can read off the (finite number of) terms which diverge once the regulator is removed. A counterterm action can then be constructed as invariant local functional of the metric and fields on the boundary of the regulated space-time in such a way that these terms are cancelled.

This type of duality can be extended to the case of arbitrary Dp -branes ($p \neq 3$), for which the world-volume gauge theory is again equivalent to the supergravity on the near-horizon background generated by the Dp -branes [16]. The dual gauge theory is a $(p+1)$ -dimensional $U(N)$ supersymmetric Yang-Mills theory. Contrarily to the case of the D3-branes, it has a dimensionful coupling constant and the effective coupling depends on the energy scale: the gauge theory is no longer conformal.

In such cases, holography has not been explored very extensively [17, 18, 19, 20, 21, 22] and just recently an exhaustive extension of the holographic renormalization procedure has been formulated [23, 24, 25]. The key point in [24] is the existence of a frame [26] in which the near-horizon geometry induced by the Dp -branes is conformally $AdS_{p+2} \times S^{8-p}$ [27, 28, 29]. In this frame, the existence of a generalized conformal symmetry [30] becomes manifest and the radial direction (transverse to the boundary) acquires the meaning of energy scale of the dual gauge theory [29, 31], as in the original AdS/CFT -correspondence. Moreover,

the holographic RG flow turns out to be trivial and the theory flows just because of the dimensionality of the coupling constant. In the case of the D4-branes, the theory flows to a 6-dimensional fixed point at strong coupling: the world-volume theory of D4-branes flows to the world-volume theory of M5-branes. In [25] an interesting observation has been made, which drastically simplifies the direct computation of the local counterterms. The $(p + 2)$ -dimensional bulk effective action, which is obtained by Kaluza-Klein reduction on the $(8 - p)$ -dimensional compact manifold, can be recovered by dimensional reduction of the theory on pure (asymptotically) $AdS_{2\sigma+1}$ on a torus. The parameter σ is related to the power of the radial direction in the dilaton and takes fractional values for some p . One can consider σ as a generic integer and, after the compactification on the torus, analytically continue it to take its actual value. A simple way to then compute the counterterm action is to map the problem to pure $AdS_{2\sigma+1}$ theory. The counterterms are therefore the ones needed to renormalize these higher dimensional pure gravity theory and it is possible to go back to the original problem by Kaluza-Klein reduction of these counterterms on a torus with a warp factor dependent on the dilaton.

Gauge/gravity correspondence can be further generalized by inserting extra degrees of freedom in the theory. More precisely, one can add a finite number of branes and consider the probe approximation, so that the backreaction on the background geometry can be neglected. Inserting probe branes introduces a fundamental hypermultiplet in the gauge theory, partially or completely breaking the original supersymmetries [32]. Also in the case of probe-brane modes, there are IR-divergences appearing and, therefore, a consistent extension of the holographic renormalization procedure is needed. Such an extension is straightforward for the case of probe-branes in a D3-brane background, but still shows a very interesting feature [33]. The probe branes wrap an $AdS_{5-k} \times S^{3-k}$ subspace of the whole $AdS_5 \times S^5$ space-time, where k is the codimension of the defect ($k = 0, 1, 2$). For $k \neq 0$, there are two different ways to describe the embedding of the probe branes: one can fix the position of the probe branes in the transverse space and study the embedding of the branes inside AdS_5 , where they wrap an AdS_{5-k} submanifold (linear embedding)¹; the other possible choice is to fix the position of the probe-branes in AdS_5 and consider the embedding of the branes in the transverse space (angular embedding)². For $k = 0$ only the latter description is possible since, in this case, the probe-branes wrap the whole AdS_5 -space. In [33] it has been shown

¹For $k = 1$ this description is non-supersymmetric and corresponds to turn on a vev for the embedding mode [34].

²Choosing this embedding corresponds in turning on a massive deformation, if $k = 0, 1$. For $k = 2$, it has been shown to be equivalent to turning on a vev deformation [33]. This is due to the fact that only in this case the mode saturates the Breitenlohner-Freedman bound [35], consequently changing the boundary expansion.

that the angular embedding mode behaves in a neighbourhood of the boundary as a free massive scalar propagating in AdS_{p+2-k} , i.e. near the boundary it has the same expansion of a free massive scalar for all the relevant orders. This implies that the counterterms needed to holographically renormalize these degrees of freedom are just the same counterterms needed for a free massive scalar in AdS -space.

In this paper we discuss holographic renormalization of probe branes in general Dp -brane backgrounds. Systems of this type have been studied especially in attempt to try to construct holographic duals of large- N QCD [36, 37, 38, 39, 40] and may be potentially interesting to infer features of condensed matter systems. We begin with the detailed analysis of the systems with a codimension-one defect and analyze both of the two classes of embeddings. The linear embedding is straightforward to treat. The action shows just a single term which diverges as the boundary is approached, and it is renormalized by a term proportional to the volume of the boundary of a warped AdS -space.

The angular embedding description is also very interesting. As for the conformal case [33], the expansion of the mode near the boundary turns out to be the same of a free massive scalar propagating in AdS . This AdS -space is higher dimensional with respect to the conformally- AdS space that the probe branes wrap. These “extra”-dimensions are due to the leading contribution of the dilaton and their number can be fractional, as in the case of the theory with no flavour [25]. We show that the DBI-action for the probe branes in Dp -branes background can be equivalently rewritten as a DBI-action in an AdS_{q+1} geometry, q being initially an arbitrary integer. From this perspective, the embedding mode behaves as a free massive scalar as mentioned earlier, and the Breitenlohner-Freedman bound is strictly satisfied for any p . The counterterms which renormalize the action are therefore the ones needed by a free massive scalar particle in AdS_{q+1} . In order to restore the original setup, we can compactify the AdS_{q+1} on a T^{q-p} torus and analytically continue q to its actual value q_p . The torus T^{q-p} has a warp factor which again depends on the dilaton. However, in this case one needs to take into account not just the dilaton factor coming from the induced metric on the world-volume, but also the one contained in the original DBI-action. We also apply the renormalized action obtained to compute the one point correlator for the operator dual to the embedding modes. These procedure can be straightforwardly extended to brane intersections with codimension-0 and codimension-2. As for the D3/D3 system [33], all the Dp/Dp systems ($p < 5$) show a different counterterm structure with respect to theories with a lower-codimension defect: this is the only case for which a term proportional to $(\log \epsilon)^{-1}$ appears. It is a direct consequence of the fact that the embedding mode saturates the Breitenlohner-Freedman bound, which also imply that the one-point correlator for the dual operator is expressed through the coefficient of the normalizable mode.

This viewpoint can be extended also to the linear embedding case treated earlier. Again, one can show the equivalence between the DBI-action of probe-branes embedded in Dp -backgrounds and the DBI-action of probe branes embedded into higher dimensional AdS -space. The only counterterm needed is proportional to the volume of the boundary of the higher dimensional AdS -space and its Kaluza-Klein reduction on a T^{q-p} returns the same counterterm found before. This case is indeed very simple by itself, so the higher-dimensional view-point is not strictly needed. However, it contributes to a more general understanding of the structure of the systems we are considering. The idea proposed in [25] can therefore be extended to the case of non-conformal systems with flavours.

The paper is organized as follows. In section 2, we briefly review the holographic renormalization procedure for general Dp -backgrounds. In section 3 we introduce the $Dp/D(p+2)$ systems and we generally discuss it taking into consideration all the possible description for the embedding of the probe $D(p+2)$ -branes. In section 4 we fix the probe branes to wrap the maximal sphere $S^2 \subset S^{8-p}$ and consider the embedding of the probe branes into the $(p+2)$ -dimensional non-compact manifold. We show that there is a single counterterm needed for the renormalization of the action and outline the first suggestions about possible relations between these systems and probe branes in a higher dimensional AdS -space. We also compute the one-point correlator. In section 5, we show that, the expansion near the boundary of the angular embedding function satisfies the equation of motion for a free massive scalar in a higher-dimensional AdS space-time for all the orders of interest and we implement the procedure outlined earlier for the computation of the counterterms. Furthermore, we compute the one-point correlator for the related boundary operator. In section 6 we extend this approach to general brane intersections. Section 7 contains conclusion and a summary of the results.

2 Holographic Renormalization of Dp -branes background

Let us start with recalling the brane solution from type IIA/IIB string theory in Euclidean signature

$$ds_{10}^2 = \left(1 + \frac{r_p^{7-p}}{r^{7-p}}\right)^{-1/2} \delta_{\mu\nu} dx^\mu dx^\nu + \left(1 + \frac{r_p^{7-p}}{r^{7-p}}\right)^{1/2} ds_{\text{T}}^2, \quad (2.1)$$

where $\mu, \nu = 0, \dots, p$, ds_{T}^2 is the line element for the transverse space and the constant r_p is defined through

$$r_p^{7-p} \stackrel{\text{def}}{=} (2\sqrt{\pi})^{5-p} \Gamma\left(\frac{7-p}{2}\right) g_s N (\alpha')^{(7-p)/2} \equiv d_p g_s N (\alpha')^{(7-p)/2}. \quad (2.2)$$

The decoupling limit

$$g_s \rightarrow 0, \quad \alpha' \rightarrow 0, \quad U \stackrel{\text{def}}{=} \frac{r}{\alpha'} \equiv \text{fixed}, \quad g_{\text{YM}}^2 N \equiv \text{fixed}, \quad (2.3)$$

where the coupling constant g_{YM} is dimensionful and defined by

$$g_{\text{YM}}^2 \stackrel{\text{def}}{=} g_s (2\pi)^{p-2} (\alpha')^{(p-3)/2}, \quad (2.4)$$

corresponds to the near horizon geometry for the Dp-branes

$$\begin{aligned} ds_{10}^2 &= g_{\text{MN}} dx^{\text{M}} dx^{\text{N}} = \\ &= \alpha' \left\{ \left(\frac{U}{U_p} \right)^{(7-p)/2} \delta_{\mu\nu} dx^\mu dx^\nu + \left(\frac{U_p}{U} \right)^{(7-p)/2} [dU^2 + U^2 d\Omega_{8-p}^2] \right\}, \end{aligned} \quad (2.5)$$

where the constant U_p has been defined as

$$U_p^{7-p} \stackrel{\text{def}}{=} \frac{d_p}{(2\pi)^{p-2}} g_{\text{YM}}^2 N, \quad (2.6)$$

while the dilaton and the background $(p+1)$ -form are respectively

$$e^\phi = \frac{g_{\text{YM}}^2 N}{(2\pi)^{p-2} N} \left(\frac{U}{U_p} \right)^{(7-p)(p-3)/4}, \quad C_{0\dots p} = \frac{(2\pi)^{p-2} (\alpha')^{(p+1)/2} N}{g_{\text{YM}}^2 N} \left(\frac{U}{U_p} \right)^{7-p}. \quad (2.7)$$

As we just mentioned, for $p \neq 3$ the coupling constant is dimensionful and, therefore, the effective coupling turns out to run with the energy scale:

$$g_{\text{eff}}^2 = g_{\text{YM}}^2 N U^{p-3}. \quad (2.8)$$

The background metric (2.5) is actually conformal to an $AdS_{p+2} \times S^{8-p}$ space for $p \neq 5$ [29, 31]. This can be easily seen by redefining the radial coordinate according to

$$\frac{u^2}{u_p^2} \stackrel{\text{def}}{=} \left(\frac{d_p}{(2\pi)^{p-2}} g_{\text{YM}}^2 N \right)^{-1} U^{5-p}, \quad u_p = \frac{5-p}{2} \quad (2.9)$$

and rewriting the line element (2.5) as

$$ds_{10}^2 = \left(N e^\phi \right)^{2/(7-p)} d\hat{s}_{10}^2, \quad (2.10)$$

so that the line element $d\hat{s}_{10}^2$ describes an $AdS_{p+2} \times S^{8-p}$ geometry in the Poincaré patch

$$d\hat{s}_{10}^2 = \hat{g}_{\text{MN}} dx^{\text{M}} dx^{\text{N}} = \alpha' B_p \left\{ u^2 \delta_{\mu\nu} dx^\mu dx^\nu + \frac{du^2}{u^2} + u_p^2 d\Omega_{8-p}^2 \right\}. \quad (2.11)$$

The near-horizon geometry of a background generated by a stack of N Dp-branes is therefore conformal to a $AdS_{p+2} \times S^{8-p}$ space with conformal factor $(N e^\phi)^{\frac{2}{7-p}}$. In the coordinates $\{x^\mu, u\}$, the dilaton writes

$$e^\phi = \frac{A_p}{N} u^{(p-3)(7-p)/2(5-p)}. \quad (2.12)$$

For the purpose of holographic renormalization, it is convenient to rewrite the metric (2.11) in Fefferman-Graham coordinates through the radial coordinate redefinition $\rho = u^{-1}$. The original ten-dimensional metric (2.5) acquires the form

$$\begin{aligned} ds_{10}^2 &= \alpha' B_p \left(N e^\phi \right)^{2/(7-p)} \left\{ \frac{\delta_{\mu\nu} dx^\mu dx^\nu + d\rho^2}{\rho^2} + u_p^2 d\Omega_{8-p}^2 \right\} \equiv \\ &\equiv \alpha' B_p \left(N e^\phi \right)^{2/(7-p)} d\tilde{s}_{10}^2, \end{aligned} \quad (2.13)$$

with

$$e^\phi = \frac{A_p}{N} \rho^{-(p-3)(7-p)/2(5-p)}. \quad (2.14)$$

For the sake of generality, let us rewrite the line element $d\tilde{s}_{10}^2$ in the following form

$$d\tilde{s}_{10}^2 = \mathbf{g}_{\tilde{\mu}\tilde{\nu}} d\tilde{x}^{\tilde{\mu}} d\tilde{x}^{\tilde{\nu}} + u_p^2 d\Omega_{8-p}^2 = \frac{\mathbf{g}_{\mu\nu}(x, \rho) dx^\mu dx^\nu + d\rho^2}{\rho^2} + u_p^2 d\Omega_{8-p}^2 \quad (2.15)$$

In [24] it was shown that in a neighbourhood of the boundary the expansion for both the metric $\mathbf{g}_{\mu\nu}(x, \rho)$ and the dilaton $\phi(x, \rho)$ may contain fractional powers of ρ :

$$\begin{aligned} \mathbf{g}_{\mu\nu}(x, \rho) &= \mathbf{g}_{\mu\nu}^{(0)} + \rho^2 \mathbf{g}_{\mu\nu}^{(2)} + \dots + \rho^{2\frac{7-p}{5-p}} \left[\mathbf{g}_{\mu\nu}^{(2\frac{7-p}{5-p})} + (\delta_{p,3} + \delta_{p,4}) \mathbf{h}_{\mu\nu}^{(2\frac{7-p}{5-p})} \log \rho \right] + \dots \\ \phi(x, \rho) &= -\frac{(p-3)(7-p)}{2(5-p)} \log \rho + \left(\varepsilon_{p,3} \frac{7-p}{p-3} + \delta_{p,3} \right) \times \\ &\times \left\{ \kappa^{(0)} + \rho^2 \kappa^{(2)} + \dots + \rho^{2\frac{7-p}{5-p}} \left[\kappa^{(2\frac{7-p}{5-p})} + (\delta_{p,3} + \delta_{p,4}) \mathbf{k}^{(2\frac{7-p}{5-p})} \log \rho \right] + \dots \right\}, \end{aligned} \quad (2.16)$$

where $\varepsilon_{p,3}$ is 0 for $p = 3$ and 1 otherwise. The undetermined coefficients of these expansions appear at order $\mathcal{O}(\rho^0)$ and $\mathcal{O}(\rho^{2\frac{7-p}{5-p}})$. Furthermore, the boundary expansions (2.16) exhibit a behaviour similar to asymptotically AdS backgrounds. More specifically, the Dp backgrounds with $p < 3$ do not show any logarithmic term, as for asymptotically AdS -spaces with even dimensions. For $p = 3, 4$, which are the only values of p for which the last term in (2.16) appears at an even integer power of ρ , they behave as for asymptotically AdS -spaces with odd dimensions. In [25], it was explicitly shown the equivalence between the effective supergravity action which allows for Dp -brane solutions (compactified on S^{8-p}) and the action for a theory on pure $AdS_{2\sigma+1}$ with cosmological constant $\Lambda = -\sigma(2\sigma - 1)$ when it is compactified on a $T^{2\sigma-d}$

$$\begin{aligned} ds_{2\sigma+1}^2 &= \mathbf{g}_{\hat{\alpha}\hat{\beta}}^{(p+2)} dx^{\hat{\alpha}} dx^{\hat{\beta}} + e^{2\frac{\phi(x,\rho)}{2\sigma-(p+1)}} \delta_{ab} dx^a dx^b \\ S &= L_{AdS} \int d^{2\sigma+1}x \sqrt{g_{(2\sigma+1)}} \left[R^{(2\sigma+1)} + 2\sigma(2\sigma - 1) \right] = \\ &= L \int d^{p+2}x e^\phi \sqrt{\mathbf{g}_{(p+2)}} \left[R^{(p+2)} + \frac{2\sigma - (p+2)}{2\sigma - (p+1)} (\partial\phi)^2 + 2\sigma(2\sigma - 1) \right], \end{aligned} \quad (2.17)$$

with the identification $L = L_{AdS} (2\pi R_T)^{2\sigma - (p+1)}$. From a $(2\sigma + 1)$ -dimensional perspective, the action (2.17) is renormalized by the standard *AdS*-counterterms. Considering 2σ as an arbitrary integer, one can perform the Kaluza-Klein reduction of the *AdS*-counterterms with the metric ansatz (2.17) and then analytically continue 2σ to take the fractional value $2\sigma_p = 2(7 - p)/(5 - p)$. The counterterms for pure *AdS*-gravity are [9]

$$S_{ct} = L_{AdS} \int_{\rho=\epsilon} d^{2\sigma} x \sqrt{\gamma_{(2\sigma)}} \left[2(2\sigma - 1) + \frac{1}{2\sigma - 2} R_{(\gamma)} + \frac{1}{(2\sigma - 4)(2\sigma - 2)^2} \left(R_{AB}^{(\gamma)} R_{(\gamma)}^{AB} - \frac{\sigma}{2(2\sigma - 1)} R_{(\gamma)}^2 \right) - a_{(2\sigma)} \log \epsilon + \dots \right], \quad (2.18)$$

with $a_{(2\sigma)}$ indicating the conformal anomaly. Dimensional reducing (2.18) on the metric ansatz (2.17), one obtain the counterterms for the $(p + 2)$ -dimensional background [24, 25]

$$S_{ct} = L \int_{\rho=\epsilon} d^{p+2} x e^\phi \sqrt{\gamma_{(p+1)}} \left\{ 2(2\sigma - 1) + \frac{1}{2\sigma - 2} \left(R_{(p+1)} + \frac{2\sigma - p - 2}{2\sigma - p - 1} (\partial\phi)^2 \right) + \delta_{\sigma, 3} \left[\frac{1}{(2\sigma - 4)(2\sigma - 2)^2} \left(R_{\alpha\beta}^{(p+1)} - 2\frac{p-3}{7-p} (\nabla_\alpha \partial_\beta \phi + \partial_\alpha \phi \partial_\beta \phi) \right)^2 + \frac{1}{(2\sigma - 4)(\sigma - 1)^2} \frac{p-3}{7-p} (\square_{(p+1)} \phi + (\partial\phi)^2)^2 - \frac{\sigma}{(2\sigma - 4)(2\sigma - 2)^2} \left(R_{(p+1)} - 4\frac{p-3}{7-p} (\square_{(p+1)} \phi + (\partial\phi)^2) \right)^2 + a_{(2\sigma)} \log \epsilon \right] \right\}. \quad (2.19)$$

The explicit expression for the conformal anomaly $a_{(2\sigma)}$ is provided in [24].

The renormalized action is then given by

$$S_{\text{ren}} = \lim_{\epsilon \rightarrow 0} [S|_\epsilon + S_{\text{GH}} + S_{\text{ct}}], \quad (2.20)$$

where S_{GH} is the standard Gibbons-Hawking term which needs to be introduced in order to have a well-defined variational principle. From an *AdS* (and therefore conformal) point of view, the one-point correlator for the boundary stress-energy tensor is [8, 9]

$$\langle T_{AB} \rangle_{(2\sigma)} = \frac{2}{\sqrt{\gamma_{(2\sigma)}}} \frac{\delta S_{\text{ren}}}{\delta \gamma_{(2\sigma)}^{\alpha\beta}}. \quad (2.21)$$

The dimensional reduction of (2.21) on the torus $T^{2\sigma - p - 1}$ returns both the one-point correlator for $(p+1)$ -dimensional boundary stress-energy tensor and the one for the scalar operator dual to the dilaton field:

$$\begin{aligned} e^{\kappa(0)} (2\pi R_T)^{2\sigma - p - 1} \langle T_{\alpha\beta} \rangle_{(2\sigma)} &= 2\sigma L e^{\kappa(0)} g_{\alpha\beta}^{(2\sigma)} = \langle T_{\alpha\beta} \rangle_{(p+1)} \\ e^{\kappa(0)} (2\pi R_T)^{2\sigma - p - 1} \langle T_{ab} \rangle_{(2\sigma)} &= \frac{4\sigma L}{2\sigma - p - 1} e^{\frac{2\sigma - p + 1}{2\sigma - p - 1} \kappa(0)} \kappa_{(2\sigma)} \delta_{ab} = -e^{\frac{2}{2\sigma - p - 1} \phi} \delta_{ab} \langle \mathcal{O}_\phi \rangle_{(p+1)} \end{aligned} \quad (2.22)$$

The remarkable observation of [25] that non-conformal backgrounds can be mapped into higher dimensional asymptotically AdS -geometries drastically simplifies the study of the dynamics of such systems, which may be determined in terms of the dynamics of conformal systems.

3 Brane intersections with codimension-1 defect

In the background (2.13) we introduce M parallel probe $D(p+2)$ -branes ($M \ll N$) according to the following intersection configuration

	0	1	2	...	p-2	p-1	p	p+1	p+2	p+3	p+4	...	8	9
Dp	X	X	X	...	X	X	X					...		
$D(p+2)$	X	X	X	...	X	X		X	X	X		...		

The probe branes wrap an internal 2-sphere $S^2 \subset S^{8-p}$. Given the presence of a codimension-1 defect, the embedding of the $D(p+2)$ branes in the Dp -brane background can in principle be described through two functions $x^p \equiv z(\rho)$ and $\theta \equiv \theta(\rho)$, where θ is one of the angular coordinates of S^{8-p} :

$$d\Omega_{8-p}^2 = d\theta^2 + \sin^2 \theta d\Omega_2^2 + \cos^2 \theta d\Omega_{5-p}^2 \quad (3.1)$$

For the moment, let us keep both of the two embedding functions and consider the pure geometrical case, in which the probe branes do not carry any gauge field. The action for the probe branes is given only by the DBI-term

$$\begin{aligned} S_{D(p+2)} &= M T_{D(p+2)} \int d^{p+3} \xi e^{-\phi} \sqrt{g_{p+3}} = \\ &= M T_{D(p+2)} \mathcal{N}_p \int d^{p+3} \xi e^{2\frac{p-2}{7-p}\phi} \sqrt{\tilde{g}_{p+3}} \end{aligned} \quad (3.2)$$

where

$$\mathcal{N}_p \stackrel{\text{def}}{=} (\alpha')^{(p+3)/2} B_p^{(p+3)/2} N^{(p+3)/(7-p)} \quad (3.3)$$

and g_{p+3} and \tilde{g}_{p+3} are the determinants of the world-volume metric induced by the background metrics g_{MN} and \tilde{g}_{MN} respectively. The price one pays in changing frame, beside the overall constant \mathcal{N}_p , is a shift in the dilaton factor of the world-volume action

$$e^{-\phi} \longrightarrow e^{2\frac{p-2}{7-p}\phi}, \quad (3.4)$$

which substantially leaves the structure of the action unchanged. It is interesting to notice that, in the frame we are now considering, there is no non-trivial dilaton dependence in (3.2) for $p = 2, 3$. The induced metric on the $D(p+2)$ -brane world-volume is

$$ds_{p+3}^2 = \frac{\delta_{\alpha\beta}}{\rho^2} dx^\alpha dx^\beta + \left[1 + (z')^2 + u_p^2 \rho^2 (\theta')^2 \right] \frac{d\rho^2}{\rho^2} + u_p^2 \sin^2 \theta d\Omega_2^2, \quad (3.5)$$

where the indices $\alpha, \beta = 0, \dots, p-1$ and the prime $'$ indicates the first derivative with respect to the radial coordinate ρ . The action (3.2) can be easily integrate over the S^2 coordinates to give

$$\begin{aligned} S_{D(p+2)} &= M T_{D(p+2)} \hat{\mathcal{N}}_p \int dt d^{p-1}x d\rho e^{2\frac{p-2}{7-p}\phi} \sin^2 \theta \sqrt{\tilde{g}_{p+1}} \\ &= M T_{D(p+2)} \hat{\mathcal{N}}_p \int dt d^{p-1}x \frac{d\rho}{\rho^{p+1}} e^{2\frac{p-2}{7-p}\phi} \sin^2 \theta \sqrt{1 + \rho^{-2} (\partial z)^2 + u_p^2 (\partial \theta)^2}, \end{aligned} \quad (3.6)$$

with $x^{\hat{\alpha}} = \{x^\alpha, \rho\}$, $\hat{\mathcal{N}}_p = u_p^2 \text{vol} \{S^2\} \mathcal{N}_p$, $(\partial f)^2 \equiv \mathbf{g}^{\hat{\alpha}\hat{\beta}} (\partial_{\hat{\alpha}} f) (\partial_{\hat{\beta}} f)$ and

$$\mathbf{g}_{\hat{\alpha}\hat{\beta}} dx^{\hat{\alpha}} dx^{\hat{\beta}} \stackrel{\text{def}}{=} \frac{\delta_{\alpha\beta} dx^\alpha dx^\beta + d\rho^2}{\rho^2}. \quad (3.7)$$

The action (3.6) depends on the linear embedding function $z(\rho)$ through its first derivative only. This implies that there is a first integral of motion c_z related to it

$$c_z = \frac{e^{2\frac{p-2}{7-p}\phi}}{\rho^{p+1}} \sin^2 \theta \frac{z'}{\sqrt{1 + \rho^{-2} (\partial z)^2 + u_p^2 (\partial \theta)^2}}. \quad (3.8)$$

The equation of motion for both the embedding functions $z(\rho)$ and $\theta(\rho)$ are

$$\begin{aligned} z'(\rho) &= c_z \frac{\rho^{\frac{11-p}{5-p}}}{\sqrt{\sin^2 \theta - c_z^2 \rho^{2\frac{11-p}{5-p}}}} \sqrt{1 + u_p^2 (\partial \theta)^2} \\ 0 &= \square \theta - \cot \theta \frac{c_z^2 \rho^{2\frac{11-p}{5-p}}}{\sin^2 \theta - c_z^2 \rho^{2\frac{11-p}{5-p}}} (\partial \theta)^2 - \frac{1}{2} \mathbf{g}^{\hat{\alpha}\hat{\beta}} (\partial_{\hat{\alpha}} \theta) \frac{\partial_{\hat{\beta}} [1 + u_p^2 (\partial \theta)^2]}{1 + u_p^2 (\partial \theta)^2} + \\ &\quad + \left[\frac{11-p}{5-p} \frac{c_z^2 \rho^{\frac{17-p}{5-p}}}{\sin^2 \theta - c_z^2 \rho^{2\frac{11-p}{5-p}}} + \frac{(p-2)(p-3)}{5-p} \right] \mathbf{g}^{\hat{\alpha}\hat{\beta}} (\partial_{\hat{\alpha}} \phi) (\partial_{\hat{\beta}} \theta) - \frac{2}{u_p^2} \cot \theta \frac{\sin^2 \theta}{\sin^2 \theta - c_z^2 \rho^{2\frac{11-p}{5-p}}} \end{aligned} \quad (3.9)$$

In (3.9), the operator \square is constructed through the metric $\mathbf{g}_{\hat{\alpha}\hat{\beta}}$. The case $c_z = 0$ corresponds to the case in which the probe branes bend in the transverse space only. Notice that at the boundary, the angular coordinate θ takes the value $\pi/2$, as one can straightforwardly see from the (3.9). In what follows, we discuss the two different classes of probe-branes embeddings separately.

4 Linear Embedding

The simplest case is provided by the description of the embedding of the probe branes through the linear coordinate $z(\rho)$. The probe branes wrap the maximal sphere $S^2 \subset S^{8-p}$ located at $\theta = \pi/2$, and the scalar $z(\rho)$ describes the embedding of the branes in the $(p+2)$ -dimensional

(conformally)-*AdS* manifold, where they wrap a $(p+1)$ -dimensional (conformally)-*AdS* sub-space. The action and the equation of motion for the scalar $z(\rho)$ can be obtained from (3.6) and (3.9) by setting $\theta = \pi/2$

$$\begin{aligned} S_{D(p+2)}^{(z)} &= M T_{D(p+2)} \hat{\mathcal{N}}_p \int dt d^{p-1} x \frac{d\rho}{\rho^{p+1}} e^{2\frac{p-2}{7-p}\phi} \sqrt{1 + \rho^{-2} (\partial z)^2}, \\ z'(\rho) &= c_z \frac{\rho^{\frac{11-p}{5-p}}}{\sqrt{1 - c_z^2 \rho^{2\frac{11-p}{5-p}}}}. \end{aligned} \quad (4.1)$$

As mentioned before, for $c_z = 0$ the embedding function z is constant and, therefore, the probe branes do not bend. For $c_z \neq 0$, the solution extends up to a maximum value for the radial coordinate ρ

$$\rho_{\max} = c^{-\frac{5-p}{11-p}}. \quad (4.2)$$

As pointed out in [33] for the analysis of the D3/D5 system³, the string turns back once reaches $\rho = \rho_{\max}$. This can be seen by expanding the solution (4.1) in a neighbourhood of ρ_{\max}

$$\begin{aligned} z'(\rho) &= \sqrt{\frac{5-p}{2(11-p)} \rho_{\max}} (\rho_{\max} - \rho)^{-1/2} + \dots \\ z(\rho) &= m + \sqrt{2\frac{5-p}{11-p} \rho_{\max}} (\rho_{\max} - \rho)^{1/2} + \dots \end{aligned} \quad (4.3)$$

Thus, a $Dp/D(p+2)$ system in the probe approximation and with the embedding of the probe $D(p+2)$ -branes parametrized by the linear coordinate $x^p \equiv z(\rho)$ is actually dual to defect theories of $D(p+2)/\bar{D}(p+2)$ separated by a finite distance proportional to ρ_{\max} .

We can now focus on divergences near the boundary. Near the boundary $\rho = 0$, the solution (4.1) has the following asymptotic expansion

$$\begin{aligned} z'(\rho) &= c_z \rho^{\frac{11-p}{5-p}} \left[1 + c_z^2 \rho^{2\frac{11-p}{5-p}} + \mathcal{O}\left(\rho^{4\frac{11-p}{5-p}}\right) \right] \\ z(\rho) &= m_0 + \frac{5-p}{2(8-p)} \rho^{2\frac{8-p}{5-p}} + \mathcal{O}\left(\rho^{2\frac{19-2p}{5-p}}\right) \end{aligned} \quad (4.4)$$

From the on-shell action

$$S_{D(p+2)}^{(z)} \Big|_{\text{on-shell}} = M T_{D(p+2)} \hat{\mathcal{N}}_p \int dt d^{p-1} x \int_{\epsilon}^{\rho_{\max}} d\rho \frac{\rho_{\max}^{\frac{11-p}{5-p}}}{\rho^{\frac{11-p}{5-p}}} \frac{1}{\sqrt{\rho_{\max}^{2\frac{11-p}{5-p}} - \rho^{2\frac{11-p}{5-p}}}}, \quad (4.5)$$

where the cut-off ϵ has been introduced to regularize the action near the boundary, it's easy to read off the divergent boundary action

$$S_{D(p+2)}^{(z)} \Big|_{\text{div}} = M T_{D(p+2)} \hat{\mathcal{N}}_p \int dt d^{p-1} x \frac{5-p}{6} \epsilon^{-\frac{6}{5-p}}. \quad (4.6)$$

³The D3/D5 system belongs to the class of theories we are considering, so the results discussed in section 5 of [33] are reproduced by setting $p = 3$.

The existence of a single divergent term implies the need of a single counterterm to renormalize the action. Such a counterterm is provided by a term proportional to the volume of the boundary (regularized with $\rho = \epsilon$)

$$S_{D(p+2)}^{(z)} \Big|_{\text{ct}} = M T_{D(p+2)} \hat{\mathcal{N}}_p \int dt d^{p-1}x \, \mathfrak{c} e^{2\frac{p-2}{7-p}\phi} \sqrt{\gamma_p|_\epsilon}, \quad (4.7)$$

with the coefficient \mathfrak{c} which can be easily computed to be

$$\mathfrak{c} = -\frac{5-p}{6}. \quad (4.8)$$

The finite on-shell action is given by the primitive function of (4.5) (which can be computed exactly) evaluated at $\rho = \rho_{\text{max}}$

$$S_{D(p+2)}^{(z)} \Big|_{\text{finite}} = M T_{D(p+2)} \hat{\mathcal{N}}_p \int dt d^{p-1}x \, \frac{p-5}{6} \rho_{\text{max}}^{-\frac{6}{5-p}} \varpi, \quad (4.9)$$

where ϖ is a finite constant. The holographic renormalization method for the action of a probe $D(p+2)$ -brane in a background generated by a stack of Dp -branes thus prescribe a single counterterm (4.7). Thus, the renormalized action is given by

$$S_{D(p+2)}^{(z)} \Big|_{\text{ren}} = \lim_{\epsilon \rightarrow 0} S_{\text{tot}}^{(z)}(\epsilon) = \lim_{\epsilon \rightarrow 0} \left[S_{D(p+2)}^{(z)} \Big|_{\text{on-shell}} + S_{D(p+2)}^{(z)} \Big|_{\text{ct}} \right] \quad (4.10)$$

We can now compute the one-point correlator of the boundary operator \mathcal{O}_z associated with the scalar function z

$$\langle \mathcal{O}_z \rangle = - \lim_{\epsilon \rightarrow 0} \frac{1}{\epsilon^{\frac{11-p}{5-p}}} \frac{z'}{\sqrt{1+(z')^2}} \Big|_\epsilon = -c_z. \quad (4.11)$$

One comment is in order. It is remarkable that the divergent action (4.6) is the same of the one that one would obtain from probe D-branes in an AdS -space of dimensions $6/(5-p)+1$ and the counterterm (4.7) is actually given in terms of the volume of the boundary of this AdS -space. We will make this observation more precise in section 7.

5 Angular Embedding

Let us now fix the position of the probe $D(p+2)$ -branes in the $(p+2)$ -dimensional (conformal)- AdS space ($z = 0$) and let us consider their embedding in the transverse space, which is parametrized by the angular coordinate $\theta(\rho)$. The action and the equation of motion can be obtain from (3.2) and (3.9) respectively by setting $c_z = 0 = z'$:

$$\begin{aligned} S_{D(p+2)}^\theta &= M T_{D(p+2)} \hat{\mathcal{N}}_p \int dt d^{p-1}x d\rho e^{\frac{(p-2)(p-3)}{5-p}\phi} \sin^2 \theta \sqrt{\mathfrak{g}} \sqrt{1 + u_p^2 (\partial\theta)^2} \\ 0 &= \square\theta - \frac{1}{2} \mathfrak{g}^{\hat{\alpha}\hat{\beta}} (\partial_{\hat{\alpha}}\theta) \frac{\partial_{\hat{\beta}} [1 + u_p^2 (\partial\theta)^2]}{1 + u_p^2 (\partial\theta)^2} + \frac{(p-2)(p-3)}{5-p} \mathfrak{g}^{\hat{\alpha}\hat{\beta}} (\partial_{\hat{\alpha}}\phi) (\partial_{\hat{\beta}}\theta) - \frac{2}{u_p^2} \cot \theta, \end{aligned} \quad (5.1)$$

where the dilaton ϕ has been rescaled by

$$\phi \longrightarrow \frac{(p-3)(7-p)}{2(5-p)}\phi \quad (5.2)$$

in order to make the dependence on $p-3$ explicit. As mentioned earlier, the dilaton term has a trivial profile for $p=2, 3$, which implies that, in a neighbourhood of the boundary, the scalar θ exhibits the behaviour of a massive free scalar propagating in AdS_{p+1} up to order ρ^Δ . As we will show later, in these cases the scalar $\theta(\rho)$ turns out to be tachyonic and the Breitenlohner-Freedman bound [35, 41] is satisfied. For $p \neq 2, 3$, the dilatonic term of (5.1) is relevant, but it does not spoil the free massive scalar behaviour of θ .

For the sake of generality, let us rewrite the metric (3.7) in the following form

$$\mathbf{g}_{\hat{\alpha}\hat{\beta}}dx^{\hat{\alpha}}dx^{\hat{\beta}} = \frac{\mathbf{g}_{\alpha\beta}dx^\alpha dx^\beta + d\rho^2}{\rho^2}. \quad (5.3)$$

The case of interest can be easily recovered by setting $\mathbf{g}_{\alpha\beta} = \delta_{\alpha\beta}$. Assuming that a power expansion is valid, the most general form for the scalar θ near the boundary is

$$\theta(x, \rho) = \frac{\pi}{2} + \hat{\theta}_1(x, \rho), \quad (5.4)$$

where the function $\hat{\theta}_1(x, \rho)$ is defined as

$$\hat{\theta}_1(x, \rho) = \rho^\alpha \theta_1(x, \rho) \equiv \rho^\alpha \left[\sum_{i=0}^{\infty} \vartheta_{\alpha i}(x) \rho^{\beta_i} + \sum_{i=0}^{\infty} \psi_{\alpha i}(x) \rho^{\beta_i} \log \rho + \sum_{i=0}^{\infty} \sum_{j=2}^s \sigma_{\beta_i, j}(x) \rho^{\beta_i} \log^j(\rho) \right] \quad (5.5)$$

It's easy to notice that, up to order $\mathcal{O}(\rho^{3\alpha})$, the equation of motion (5.1) reduces to

$$0 = \square \hat{\theta}_1 + \frac{2}{u_p^2} \hat{\theta}_1 + \frac{(p-2)(p-3)}{5-p} \mathbf{g}^{\hat{\alpha}\hat{\beta}} (\partial_{\hat{\alpha}} \phi) (\partial_{\hat{\beta}} \hat{\theta}_1) + \mathcal{O}(\rho^{3\alpha}). \quad (5.6)$$

For the cases $p=2, 3$, it's easy to see that, indeed up to order $\mathcal{O}(\rho^{3\alpha})$, the equation of motion (5.6) reduces to the equation of motion for a free massive scalar particle propagating in a background described by the metric $\mathbf{g}_{\hat{\alpha}\hat{\beta}}$ (which becomes AdS_{p+1} once one sets $\mathbf{g}_{\alpha\beta} = \delta_{\alpha\beta}$):

$$0 = \square \hat{\theta}_1 + \frac{2}{u_p^2} \hat{\theta}_1 + \mathcal{O}(\rho^{3\alpha}) \quad p=2, 3, \quad (5.7)$$

the mass M of the scalar particle being

$$M^2 = -\frac{2}{u_p^2} \equiv -\frac{8}{(5-p)^2}. \quad (5.8)$$

The AdS/CFT correspondence relates the mass (5.8) to the dimension of the dual boundary operator \mathcal{O}_θ by

$$M^2 = \Delta(\Delta - p), \quad (5.9)$$

and the solution for $\hat{\theta}_1$ can be written as

$$\hat{\theta}_1 = \rho^{\Delta_-} (\vartheta_0(x) + \dots) + \rho^{\Delta_+} (\vartheta_{(\Delta_+ - \Delta_-)}(x) + \dots), \quad (5.10)$$

where

$$\Delta_{\pm} = \frac{p}{2} \pm \frac{1}{2} \sqrt{p^2 - \frac{32}{(5-p)^2}}, \quad (5.11)$$

and the conformal dimension of \mathcal{O}_θ is $\Delta = \Delta_+$. The expression for the squared mass (5.8) clearly implies that the scalar θ is a tachyon. It's easy to check that the Breitenlohner-Freedman bound is satisfied for the cases of interests: the system is then stable and the power α in the boundary expansion (5.4) turns out to be $\alpha = \Delta_-^4$.

For $p \neq 2, 3$, the equation (5.6) at the leading order reduces to

$$M^2 = \alpha(\alpha - q_p). \quad (5.12)$$

Notice that it has the same structure of (5.9) and can be obtained from it by a simple shift

$$p \longrightarrow p + \frac{(p-2)(p-3)}{5-p} = \frac{6}{5-p} \equiv q_p. \quad (5.13)$$

The non-normalizable and normalizable modes therefore are

$$\hat{\theta}_1(x, \rho) = \rho^{\alpha_-} (\vartheta_0(x) + \dots) + \rho^{\alpha_+} (\vartheta_{(\alpha_+ - \alpha_-)}(x) + \dots), \quad (5.14)$$

with

$$\alpha \equiv \alpha_- = \frac{2}{5-p}, \quad \alpha_+ = \frac{4}{5-p}. \quad (5.15)$$

It is possible to observe from (5.1) by a simple counting that the terms in the boundary expansion of θ contributes to the divergences of the action (5.1) up to the order $\mathcal{O}(\rho^{\alpha+\beta_i})$, with $\alpha + \beta_i < (11-p)/2(5-p)$:

$$\begin{aligned} S_{D(p+2)}^\theta &= M T_{D(p+2)} \hat{\mathcal{N}}_p \int dt d^{p-1}x d\rho \frac{e^{\frac{(p-2)(p-3)}{5-p} \kappa(x, \rho)}}{\rho^{\frac{11-p}{5-p}}} \sqrt{g} \times \\ &\times \left\{ 1 + \frac{1}{2} \left[u_p^2 (\partial \hat{\theta}_1)^2 - 2 \hat{\theta}_1^2 \right] + \frac{\hat{\theta}_1^4}{3} - \frac{u_p^2}{2} \hat{\theta}_1^2 (\partial \hat{\theta}_1)^2 - \frac{u_p^4}{8} \left[(\partial \hat{\theta}_1)^2 \right]^2 + \dots \right\}. \end{aligned} \quad (5.16)$$

As a consequence, those terms which receive contribution from the second term in the equation of motion (5.1) contribute to the finite part of the action and, therefore, the solution of equation (5.6) provides all the terms of interest.

⁴More precisely, inserting the leading order of (5.4) in (5.7) and α satisfies the equation (5.9). The solutions $\alpha = \Delta_-$ and $\alpha = \Delta_+$ are the powers of ρ for the non-normalizable and normalizable modes respectively.

At leading order, the equation of motion for θ can be again written as the equation of motion for a free massive scalar and the effect of the dilatonic term is just to mimic a shift of the dimensions of the defect as in (5.13). As we observed earlier, the terms of the boundary expansion of θ , which contributes to the divergent part of the world-volume action, can be obtained by just solving (5.6). This equation can actually be reduced to the equation of a free massive scalar propagating in an AdS background with fractional dimensions. One can think to reduce the problem to D-branes wrapping an $AdS_{q+1} \times S^2$ space-time: the mode θ behaves as a massive free scalar propagating in an AdS_{q+1} geometry and, similarly to [25], the counterterms for the holographic renormalization can be simply obtained by performing a Kaluza-Klein reduction on a T^{q-p} and then analytically continuing q to take the fractional value q_p . From an AdS_{q+1} perspective, the dual boundary operator \mathcal{O}_θ has conformal dimension

$$\Delta_{q_p} = \frac{q_p}{2} + \frac{1}{2}\sqrt{q_p^2 + 4M^2}, \quad (5.17)$$

and the Breitenlohner-Freedman bound

$$q_p^2 \geq -4M^2 = \frac{32}{(5-p)^2}. \quad (5.18)$$

is satisfied for any p . Notice that q (q_p) has the natural interpretation of the number of dimensions of the boundary of the asymptotically AdS space.

Furthermore, up to the order of interest, the expansion (5.5) satisfies the equation of motion if and only if all the coefficients for the logarithms with high powers are zero: $\sigma_{i,j}(x) = 0 \ \forall i \in [2/(5-p), 4/(5-p)]$, $j \in [0, s]$. We can write the explicit expression for the solution

$$\theta_1(x, \rho) = \rho^{\frac{2}{5-p}} \left[\vartheta_0(x) + \rho^{\frac{2}{5-p}} \left(\vartheta_{(\frac{2}{5-p})}(x) + \psi_{(\frac{2}{5-p})}(x) \log \rho \right) \right] + \mathcal{O}\left(\rho^{\frac{6}{7-p}}\right), \quad (5.19)$$

with logarithms appearing at higher order and these higher orders not contributing to the divergence of the action. We can check that the expansion (5.19) coincides with the result of [33] when the D3/D5 system is considered⁵.

The equation of motion also tells us that the function $\psi_{\frac{2}{5-p}}(x)$ is non-zero only for $p = 4$ and it is determined in terms of $\vartheta_0(x)$, $\mathbf{g}_{(0)}(x)$, $\kappa_{(0)}(x)$ and $\kappa_{(2)}(x)$

$$\psi_{(\frac{2}{5-p})} = -\delta_{p,4} \frac{5-p}{2} \left[\dot{\square}_{(0)} \vartheta_{(0)} + \frac{(p-2)(p-3)}{5-p} \gamma \left[\mathbf{g}_0^{\alpha\beta} (\partial_\alpha \kappa_{(0)}) (\partial_\beta \vartheta_{(0)}) + \frac{4}{5-p} \kappa_{(2)} \vartheta_{(0)} \right] \right], \quad (5.20)$$

where $\dot{\square}_{(0)}$ denotes the d'alambertian operator defined through the metric $\mathbf{g}_{\alpha\beta}^{(0)}(x)$

⁵One can straightforwardly compare our expansion (5.19), with equations (3.6) and (3.7) of [33] setting the parameters m and n to 4 and 2 respectively.

5.1 Counterterms

In the previous section we showed that the equation of motion for the embedding function θ is satisfied, up to order $\mathcal{O}(r^\Delta)$, by a solution for a free massive scalar. The divergences in the action should therefore be the ones of a free scalar, which can be easily checked from (5.16).

Now, we claim that, in a similar fashion of [25], the counterterms for the holographic renormalization can be computed by reducing the problem to the study of D-branes which wrap an $AdS_{q+1} \times S^2$, so that the counterterms are the same as the ones needed for a massive scalar propagating in AdS_{q+1} . Then, one can perform a Kaluza-Klein reduction on a T^{q-p} and finally analytically continue q to the fractional value q_p .

Let us illustrate the procedure in more detail. Consider the DBI action

$$S_{\text{DBI}} = M T_q \hat{\mathcal{N}}_q \int dt d^{q-1} x d\rho \sin^2 \theta \sqrt{\mathfrak{g}_{(q+1)}} \sqrt{1 + u_p^2 (\partial\theta)^2}, \quad (5.21)$$

with $\mathfrak{g}_{(q+1)}$ being the determinant of the AdS_{q+1} metric. Let us now compactify the AdS_{q+1} space-time on a T^{q-p} :

$$ds_{q+3}^2 \equiv \mathfrak{g}_{AB}^{(q+1)} dx^A dx^B = \mathfrak{g}_{\alpha\beta}^{(p+1)} dx^\alpha dx^\beta + e^{2\frac{(p-2)(p-3)}{(5-p)(q-p)}\phi} \delta_{ab} dx^a dx^b, \quad (5.22)$$

where ϕ depends on the coordinates $\{x^\alpha\}_{\alpha=0}^p$ only. With such an ansatz, the DBI-action (5.21) becomes

$$S_{\text{DBI}} = M T_q \hat{\mathcal{N}}_q (2\pi R_T)^{q-p} \int dt d^{p-1} x d\rho e^{\frac{(p-2)(p-3)}{(5-p)}\phi} \sin^2 \theta \sqrt{\mathfrak{g}_{(p+1)}} \sqrt{1 + u_p^2 (\partial\theta)^2}, \quad (5.23)$$

which is equivalent to (5.1), with

$$T_q \hat{\mathcal{N}}_q (2\pi R_T)^{q-p} = T_{D(p+2)} \hat{\mathcal{N}}_p. \quad (5.24)$$

The solution (5.19) is indeed solution for the equation of motion from (5.23). One can infer that, as anticipated, the counterterms are the same needed for a massive scalar propagating in AdS_{q+1} :

$$\begin{aligned} S_{\text{ct}}^{(q)} = & M T_q \hat{\mathcal{N}}_q \int dt d^{q-1} x \sqrt{\gamma|_\epsilon} e^{\frac{(p-2)(p-3)}{5-p}\phi(x,\epsilon)} \left\{ -\frac{1}{q} + \left[\frac{\delta_{q,2}}{4} \log \epsilon - \frac{\varepsilon_{q,2}}{2q(q-1)(q-2)} \right] R_\gamma^{(q)} + \right. \\ & + \frac{\delta_{q,4}}{32} \log \epsilon \left(R_\gamma^{AB} R_{AB}^\gamma - \frac{1}{3} R_\gamma^2 \right) + \left[\frac{q-\Delta}{2} + \frac{\delta_{q,2\Delta}}{2 \log \epsilon} + \frac{q-\Delta}{2(q-1)} \left(-\frac{\delta_{q,2\Delta-2}}{2} \log \epsilon \right. \right. \\ & \left. \left. + \frac{\varepsilon_{q,2\Delta-2}}{2(2\Delta-q-2)} \right) R_\gamma \right] u_p^2 \hat{\theta}_1(x, \epsilon)^2 + \\ & \left. + \left[\frac{\varepsilon_{q,2\Delta-2}}{2(2\Delta-q-2)} - \frac{\delta_{q,2\Delta-2}}{2} \log \epsilon \right] u_p^2 \hat{\theta}_1(x, \epsilon) \square_\gamma^{(q)} \hat{\theta}_1(x, \epsilon) \right\}, \end{aligned} \quad (5.25)$$

where $\varepsilon_{p,l}$ is 0 if $p = l$ and 1 otherwise. Compactifying on the T^{q-p} , the curvature scalar $R_\gamma^{(q)}$ and the d'alambertian operator $\square_\gamma^{(q)}$ reduce to

$$\begin{aligned} R_\gamma^{(q)} &= R_\gamma^{(p)} - 2 \frac{(p-2)(p-3)}{5-p} \square_p \phi - \frac{(p-2)^2(p-3)^2}{(5-p)^2} \frac{q-p+1}{q-p} (\partial\phi)^2 \\ \square_\gamma^{(q)} &= \square_\gamma^{(p)} + \frac{(p-2)(p-3)}{5-p} \gamma^{\hat{\alpha}\hat{\beta}} (\partial_{\hat{\alpha}}\phi) \partial_{\hat{\beta}} \end{aligned} \quad (5.26)$$

Inserting (5.26) in (5.25) and analytically continuing q to q_p (the explicit expression of q_p is in (5.13)), we obtain the counterterms we were looking for. Notice that, once the analytic continuation to q_p is performed, one can realize that the first two terms in the second line of (5.25) do not contribute for any p given that $\delta_{q,4}$ would be satisfied just for $p = 7/2$. Similarly, the term proportional to $(\log \epsilon)^{-1}$ does not appear for any values of p of interest (the constraint $q - 2\Delta = 0$ is never satisfied). The counterterm action can be therefore written as

$$\begin{aligned} S_{\text{ct}}^{(\theta)} &= M T_{\text{D}(p+2)} \hat{\mathcal{N}}_p \int dt d^{p-1}x \sqrt{\gamma|_\epsilon} e^{\frac{(p-2)(p-3)}{5-p} \phi(x,\epsilon)} \times \\ &\times \left\{ -\frac{5-p}{6} + \left[\frac{\delta_{p,2}}{4} \log \epsilon - \varepsilon_{p,2} \frac{(5-p)^3}{24(p+1)(p-2)} \right] \left[R_\gamma - 2 \frac{(p-2)(p-3)}{5-p} \nabla^2 \phi - \right. \right. \\ &- \frac{(p-2)(p-3)(p^2-6p+11)}{(5-p)^2} (\partial\phi)^2 \left. \right] + \left[\frac{1}{5-p} + \frac{1}{p+1} \left(-\frac{\delta_{p,4}}{2} \log \epsilon + \frac{\varepsilon_{p,4}}{4} \frac{5-p}{p-4} \right) \times \right. \\ &\times \left(R_\gamma - 2 \frac{(p-2)(p-3)}{5-p} \nabla^2 \phi - \frac{(p-2)(p-3)(p^2-6p+11)}{(5-p)^2} (\partial\phi)^2 \right) \left. \right] u_p^2 \hat{\theta}_1(x, \epsilon)^2 + \\ &+ \left[\frac{\varepsilon_{p,4}}{4} \frac{5-p}{p-4} - \frac{\delta_{p,4}}{2} \log \epsilon \right] u_p^2 \hat{\theta}_1(x, \epsilon) \left[\square_\gamma^{(p)} \theta(x, \epsilon) + \frac{(p-2)(p-3)}{5-p} \gamma^{\alpha\beta} (\partial_\alpha \phi) (\partial_\beta \hat{\theta}_1) \right] \left. \right\}, \end{aligned} \quad (5.27)$$

It is easy to notice that for $p = 3$, the counterterm action (5.27) coincides with the one discussed in [33].

5.2 One-point correlator

We can use the holographic renormalized action obtained in the previous section to compute the one-point correlator for the operator \mathcal{O}_θ dual to the mode θ which describes the embedding of the $\text{D}(p+2)$ -branes. The standard AdS/CFT correspondence prescribes the one-point correlator to be

$$\langle \mathcal{O}_\theta \rangle_{(q)} = \lim_{\epsilon \rightarrow 0} \frac{1}{\epsilon^\Delta} \frac{1}{\sqrt{\gamma|_\epsilon}} \frac{\delta S_{\text{DBI}}^{(\theta)} \Big|_{\text{ren}}}{\delta \theta(x, \epsilon)}. \quad (5.28)$$

For the case of branes embedded in a background generated by a stack of $\text{D}p$ -branes, the prescription (5.28) can be still used, with a little subtlety. As we showed in the previous section, one can map the problem to the study of branes which wrap an $AdS_{q+1} \times S^2$ subspace.

At this level, one can apply the standard prescription (5.28) and then consider the metric ansatz (5.22). The final step is the analytic continuation of q to q_p . Considering also the relation (5.23) between the action in $(q+1)$ -dimensional asymptotically AdS space (5.21) and the original one (5.1), one can infer the relation between the one-point correlator (5.28) and the one in p -dimensions

$$\langle \mathcal{O}_\theta \rangle_{(p)} = e^{2\frac{p-2}{5-p}(\delta_{p,3} + \frac{7-p}{p-3}\varepsilon_{p,3})\kappa_{(0)}} (2\pi R_T)^{q-p} \langle \mathcal{O}_\theta \rangle_{(q)} \quad (5.29)$$

Let us analyze this procedure in detail. Applying the prescription (5.28) to the action (5.21), whose counterterms are (5.25), one obtain:

$$\langle \mathcal{O}_\theta \rangle_{(q)} = \lim_{\epsilon \rightarrow 0} \frac{1}{\epsilon^{\frac{4}{5-p}}} \frac{\epsilon^p e^{-\frac{(p-2)(p-3)}{5-p}\phi(x,\epsilon)}}{\sqrt{g(x,\epsilon)}} \left[\frac{\delta S_{\text{DBI}}^{(\theta)} \Big|_{\text{on-shell}}}{\delta\theta(x,\epsilon)} + \frac{\delta S_{\text{DBI}}^{(\theta)} \Big|_{\text{ct}}}{\delta\theta(x,\epsilon)} \right]. \quad (5.30)$$

Let us focus on the first term of (5.30), whose explicit expression is

$$\begin{aligned} \frac{1}{\epsilon^{\frac{4}{5-p}}} \frac{\epsilon^p e^{-\frac{(p-2)(p-3)}{5-p}\phi(x,\epsilon)}}{\sqrt{g(x,\epsilon)}} \frac{\delta S_{\text{DBI}}^{(\theta)} \Big|_{\text{on-shell}}}{\delta\theta(x,\epsilon)} &= -\frac{u_p^2}{\varepsilon^{\frac{2}{5-p}}} \frac{2}{5-p} \vartheta_{(0)} - \\ &- u_p^2 \left[\frac{4}{5-p} \vartheta_{(\frac{2}{5-p})} + \psi_{(\frac{2}{5-p})} \left(1 + \frac{4}{5-p} \log \varepsilon \right) \right] + \mathcal{O}\left(\rho^{\frac{2}{5-p}}\right). \end{aligned} \quad (5.31)$$

There are only two type of divergences: one of order $\mathcal{O}\left(\epsilon^{-\frac{2}{5-p}}\right)$ and the other one is a logarithm which is present just for the case of $p=4$. From (5.25), it is straightforward to notice that only the terms θ^2 and $\theta \square_\gamma^{(q)} \theta$ contribute to renormalize the correlator $\langle \mathcal{O}_\theta \rangle$. Let us now consider the contribution from the counterterm action (5.25):

$$\begin{aligned} \frac{1}{\epsilon^{\frac{4}{5-p}}} \frac{e^{-\frac{(p-2)(p-3)}{5-p}\phi(x,\epsilon)}}{\sqrt{g(x,\epsilon)}} \frac{\delta S_{\text{DBI}}^{(\theta)} \Big|_{\text{ct}}}{\delta\theta(x,\epsilon)} &= \frac{u_p^2}{\epsilon^{\frac{4}{5-p}}} \left[\frac{2}{5-p} + \frac{2}{p+1} \left(-\frac{\delta_{p,4}}{2} \log \epsilon + \frac{\varepsilon_{p,4}}{4} \frac{5-p}{p-4} \right) \times \right. \\ &\times \left(R_\gamma - 2 \frac{(p-2)(p-3)}{5-p} \square_\gamma^{(p)} \phi - \frac{(p-2)(p-3)(p^2-6p+11)}{(5-p)^2} (\partial\phi)^2 \right) \Big] \hat{\theta}_1(x,\epsilon) + \\ &+ \left[\frac{\varepsilon_{p,4}}{4} \frac{5-p}{p-4} - \frac{\delta_{p,4}}{2} \log \epsilon \right] \left[\square_\gamma^{(p)} \hat{\theta}_1(x,\epsilon) + \frac{(p-2)(p-3)}{5-p} \gamma^{\alpha\beta} (\partial_\alpha \phi) (\partial_\beta \hat{\theta}_1) \right] \end{aligned} \quad (5.32)$$

Notice that the terms containing the scalar curvature R_γ , the d'alambertian operator $\square_\gamma^{(p)}$ and the derivatives of the dilaton contribute just for $p=4$, since the leading order of these terms is $\mathcal{O}\left(\epsilon^{\frac{2}{5-p}+2}\right)$ and therefore they contribute to the correlator at order $\mathcal{O}\left(\epsilon^{\frac{2}{5-p}}\right)$. Since these terms are relevant for $p=4$ only, they contribute just to cancel the logarithmic divergence which arises in (5.31).

Taking into account the contributions (5.31) and (5.32), the one-point correlator for the operator \mathcal{O}_θ is

$$\begin{aligned} \langle \mathcal{O}_\theta \rangle_{(p)} = & -e^{2\frac{p-2}{5-p}(\delta_{p,3} + \frac{7-p}{p-3}\varepsilon_{p,3})\kappa_{(0)}} u_p^2 \left\{ \frac{2}{5-p} \vartheta_{(\frac{2}{5-p})} - \delta_{p,4} \frac{5-p}{2} \left[\dot{\square}_{(0)} \vartheta_{(0)} + \right. \right. \\ & \left. \left. + \frac{(p-2)(p-3)}{5-p} \gamma \left[\mathbf{g}_0^{\alpha\beta} (\partial_\alpha \kappa_{(0)}) (\partial_\beta \vartheta_{(0)}) + \frac{4}{5-p} \kappa_{(2)} \vartheta_{(0)} \right] \right] \right\} \end{aligned} \quad (5.33)$$

Notice that (5.33) correctly reproduces the result of [33] for the D3/D5-system once p is set to 3.

6 Brane intersections with codimension- k defect

In this section we extend the previous discussion to systems with a codimension- k defect, with $k = 0, 2$. The supersymmetric systems of interests are therefore $Dp/D(p+4)$ (codimension-0 defect) and Dp/Dp (codimension-2 defect). Generally speaking, the probe branes wrap an internal $(3-k)$ -sphere S^{3-k} and the action and the equation of motion for the embedding mode θ can be easily obtained from (5.1) by mapping $\sin^2 \theta$ to $\sin^{3-k} \theta$ and $2 \cot \theta / u_p^2$ to $(3-k) \cot \theta / u_p^2$ respectively, and $(p-2) \rightarrow (p-(k+1))$ in the dilaton factor of both the action and the equation of motion:

$$\begin{aligned} S_{D(p+4-2k)}^\theta = & M T_{D(p+4-2k)} \hat{\mathcal{N}}_{\{p,k\}} \int dt d^{p-k} x \rho e^{\frac{(p-(k+1))(p-3)}{5-p} \phi} \sin^{3-k} \theta \sqrt{\mathbf{g}} \sqrt{1 + u_p^2 (\partial \theta)^2} \\ 0 = & \square \theta - \frac{1}{2} \mathbf{g}^{\hat{\alpha}\hat{\beta}} (\partial_{\hat{\alpha}} \theta) \frac{\partial_{\hat{\beta}} [1 + u_p^2 (\partial \theta)^2]}{1 + u_p^2 (\partial \theta)^2} + \frac{(p-(k+1))(p-3)}{5-p} \mathbf{g}^{\hat{\alpha}\hat{\beta}} (\partial_{\hat{\alpha}} \phi) (\partial_{\hat{\beta}} \theta) - \frac{3-k}{u_p^2} \cot \theta, \end{aligned} \quad (6.1)$$

Using the same boundary expansion (5.5) for the embedding function, the equation of motion at the leading order gives the squared-mass relation (5.12), with the replacement

$$q_p \rightarrow q_{\{p,k\}} = p + \frac{(p-(k+1))(p-3)}{5-p} - k + 1 = 2 \frac{4-k}{5-p}, \quad (6.2)$$

The non-normalizable and normalizable modes are

$$\hat{\theta}_1 = \rho^{\alpha_-} (\vartheta_0 + \dots) + \rho^{\alpha_+} (\vartheta_{(\alpha_+ - \alpha_-)} + \dots), \quad (6.3)$$

with

$$\alpha_- = \frac{2}{5-p}, \quad \alpha_+ = 2 \frac{3-k}{5-p}. \quad (6.4)$$

The boundary expansion (5.5) is constrained by the equation of motion (6.1) to have $\sigma_{i,j}(x) = 0$, $\forall j \in [2, s]$, $\forall i \in [\alpha_-, \alpha_+]$. The terms in the expansion (5.5) contributes to the divergences up to the order $\mathcal{O}(\alpha + \beta_i)$ with $\alpha + \beta_i < [(13-2k)-p]/2(5-p)$. Up to the order of interest,

there are no higher power logarithm in the solution for the embedding mode and this coincides with the solution for a massive free particle in a $(q_{\{p,k\}} + 1)$ -dimensional asymptotically AdS -space. From this AdS -perspective, the dual operator \mathcal{O}_θ has conformal dimension $\Delta = \alpha_+$. The correct counterterms can be obtained again considering branes $(q + 1)$ -dimensional asymptotically AdS -space, for which the counterterms are given in (5.25) (after replacing $(p-2) \rightarrow (p-k-1)$ in the dilaton factor), dimensional reducing them on a torus $T^{q-p+k-1}$, and analytically continuing q to $q_{\{p,k\}}$. Notice that for codimension-2 defects, the Breitenlohner-Freedman bound is saturated and the solution of the equation of motion acquires the following form

$$\hat{\theta}_1(x, \rho) = \rho^{\frac{2}{5-p}} [\vartheta_0(x) + \psi_0(x) \log \rho + \dots]. \quad (6.5)$$

This is the only case in which the term proportional to $(\log \epsilon)^{-1}$ appears. The one-point correlator $\langle \mathcal{O}_\theta \rangle_{(p-k)}$ is

$$\begin{aligned} \langle \mathcal{O}_\theta \rangle_{(p-k)} &= e^{2\frac{p-k-1}{5-p} \left(\delta_{p,3} + \frac{7-p}{p-3} \varepsilon_{p,3} \right) \kappa_{(0)}} (2\pi R_T)^{q-p+k-1} \langle \mathcal{O}_\theta \rangle_{(q)} = \\ &= e^{2\frac{p-k-1}{5-p} \left(\delta_{p,3} + \frac{7-p}{p-3} \varepsilon_{p,3} \right) \kappa_{(0)}} (2\pi R_T)^{q-p+k-1} \lim_{\epsilon \rightarrow 0} \frac{\varepsilon_{k,2} + \delta_{k,2} \log \epsilon}{\epsilon^\Delta} \frac{1}{\sqrt{\gamma|_\epsilon}} \frac{\delta S_{DBI}^{(\theta)}|_{\text{ren}}}{\delta \theta(x, \epsilon)} \end{aligned} \quad (6.6)$$

For the supersymmetric case with $k = 2$, one obtains

$$\langle \mathcal{O}_\theta \rangle_{(p-2)} = \vartheta_{(0)}. \quad (6.7)$$

This means that in the Dp/Dp the one-point correlator for the operator dual to the embedding mode is determined by the coefficient of the normalizable mode: the observation made in [33] that the brane separation appears as a vev for the D3/D3-system extends to all the other systems with a codimension-2 defect ($p < 5$).

7 Conclusion

In this paper, we extended the holographic renormalization method to probe D-branes in non-conformal backgrounds. The key observation is that, as for theories with no flavours [25], the computation can be reduced to the computation of counterterms for probe branes in (higher-dimensional) asymptotically AdS space-times. More specifically, the mode which describes the embedding of the probe branes behaves as a free massive scalar propagating in a higher-dimensional AdS space-time, at least in a neighbourhood of the boundary and for all the orders which contribute to the divergent terms of the action. The enhancement of the number of dimensions is a direct consequence of the presence of a non-trivial profile for the dilaton. We explicitly showed that the DBI-action for the probe D-branes in non-conformal

backgrounds is equivalent to the DBI-action for probe branes in a higher-dimensional AdS -space: the original form for the DBI-action can be recovered by a Kaluza-Klein reduction on a warped torus $T^{q-p+k-1}$, where the warped factor depends on the dilaton field. Strictly speaking, the extra number of dimensions is fractional, so this picture has been made useful by considering number of dimensions of this AdS -space as an arbitrary integer and then performing analytical continuation to the actual fractional value after the computation. From this higher dimensional AdS view-point, we observed that the angular embedding mode strictly satisfies the Breitenlohner-Freedman bound, except for Dp/Dp systems for which the bound is saturated. For the latter class of systems, the one-point correlator is expressed in terms of the coefficient of the normalizable mode. This perspective drastically simplifies the computation of the counterterms for the holographic renormalization: they are just given by the counterterms for a massive scalar particle propagating in this AdS -space. Furthermore, it allows to straightforwardly apply the standard AdS/CFT prescription for the computation of one-point correlators.

One can extend this view-point also to the simple case of the linear embedding. In section 4, we easily computed the single counterterm needed by inspecting the only divergent term of the action in a neighbourhood of the boundary. We also argued that the only divergent term appearing in the action has the same behaviour of the one that one would obtain in the action of branes in a “ $6/(5-p)+1$ ”-dimensional AdS -space. More precisely, it is easy to see that DBI-action of branes in AdS_{q+1} , q being again an arbitrary integer,

$$S_{\text{DBI}}^{(z)} = M T_q \hat{\mathcal{N}}_q \int dt d^{q-1} x d\rho \sqrt{\mathfrak{g}_{(q+1)}} \sqrt{1 + \rho^{-2} (\partial z)^2} \quad (7.1)$$

reduces to (4.1) if a Kaluza-Klein reduction is performed on the metric ansatz (5.22) and then q is analytically continued to $6/(5-p)$, and (4.4) is the solution of the equation of motion from (7.1), for $\mathfrak{g}_{\alpha\beta} = \delta_{\alpha\beta}$. Inserting the solution (4.4) in the action (7.1), one can see that the only divergent term comes from the volume of AdS_{q+1} , for which the counterterms are well-known [13]. For $\mathfrak{g}_{\alpha\beta} = \delta_{\alpha\beta}$, only the term proportional to the volume of the boundary of AdS_{q+1} contributes.

Even if the simplicity of the linear description does not explicitly require any different perspective, it is useful to have a consistent and completely general AdS -viewpoint.

The remarkable observation of [25] that non-conformal backgrounds can be mapped into higher dimensional asymptotically AdS -geometries drastically simplifies the study of the dynamics of such systems, which may be determined in terms of the dynamics of conformal systems. We showed that also the degrees of freedom that can be introduced by adding probe branes may behave as degrees of freedom in asymptotically AdS space-times and, therefore, it may be possible to determine all the physics of these systems in terms of known results for the conformal case.

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